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KAN EXTENSIONS ALONG PROMONOIDAL FUNCTORS

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Abstract. Strong promonoidal functors are defined. Left Kan extension (also called "existential quantification") along a strong promonoidal functor is shown to be a strong monoidal functor. A construction for the free monoidal category on a promonoidal category is provided. A Fourier-like transform of presheaves is defined and shown to take convolution product to cartesian product.

Let $\mathcal{V}$ be a complete, cocomplete, symmetric, closed, monoidal category. We intend that all categorical concepts throughout this paper should be $\mathcal{V}$-enriched unless explicitly declared to be "ordinary". A reference for enriched category theory is [10], however, the reader unfamiliar with that theory can read this paper as written with $\mathcal{V}$ the category of sets and $\otimes$ for $\mathcal{V}$ as cartesian product; another special case is obtained by taking all categories and functors to be additive and $\mathcal{V}$ to be the category of abelian groups. The reader will need to be familiar with the notion of promonoidal category (used in [2], [6], [3], and [1]): such a category $\mathcal{A}$ is equipped with functors $P : \mathcal{A}^{op}\otimes\mathcal{A}^{op}\otimes\mathcal{A}\longrightarrow\mathcal{V}$, $J : \mathcal{A}\longrightarrow\mathcal{V}$, together with appropriate associativity and unit constraints subject to some axioms. Let $\mathcal{C}$ be a cocomplete monoidal category whose tensor product preserves colimits in each variable. If $\mathcal{A}$ is a small promonoidal category then the functor category $[\mathcal{A}, \mathcal{C}]$ has the *convolution* monoidal structure given by

$$F{*}G = \int^{A,A'} P(A, A', -)\otimes(FA\otimes GA')$$

(see [7], Example 2.4).

Suppose $\mathcal{A}$ and $\mathcal{B}$ are promonoidal categories. A *promonoidal functor* is a functor $\Phi : \mathcal{A}\longrightarrow\mathcal{B}$ together with natural transformations

$$\phi_{AA'A''} : P(A, A', A'')\longrightarrow P(\Phi A, \Phi A', \Phi A''), \qquad \phi_A : JA\longrightarrow J\Phi A$$

satisfying two axioms; see [2], [5] for details. When $\mathcal{A}$, $\mathcal{B}$ are small it means that the functor

$$[\Phi, 1] : [\mathcal{B}, \mathcal{V}]\longrightarrow[\mathcal{A}, \mathcal{V}]$$

is canonically (via the natural transformations ϕ) a monoidal functor in the sense of [8]. In particular, if $\mathcal{A}$, $\mathcal{B}$ are monoidal categories, promonoidal functors $\Phi : \mathcal{A}\longrightarrow\mathcal{B}$ are precisely monoidal functors.

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Our purpose here is to define and discuss "existential quantification" along promonoidal functors. For any promonoidal functor $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$, the natural transformations

$$P(A, A', A'') \otimes \mathcal{B}(\Phi A'', B) \xrightarrow{\phi \otimes 1} P(\Phi A, \Phi A', \Phi A'') \otimes \mathcal{B}(\Phi A'', B) \xrightarrow{\mu} P(\Phi A, \Phi A', B)$$

$$JA \otimes \mathcal{B}(\Phi A, B) \xrightarrow{\phi \otimes 1} J\Phi A \otimes \mathcal{B}(\Phi A, B) \xrightarrow{\mu} JB$$

(where the arrows μ are part of the functoriality of P, J) induce natural transformations

$$\int^{A''} P(A, A', A'') \otimes \mathcal{B}(\Phi A'', B) \xrightarrow{\rho} P(\Phi A, \Phi A', B)$$

$$\int^{A} JA \otimes \mathcal{B}(\Phi A, B) \xrightarrow{\rho} JB.$$

We call $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ *strong* when these arrows ρ are all invertible. In particular, when $\mathcal{A}$, $\mathcal{B}$ are monoidal, strong promonoidal amounts to strong monoidal (= tensor-and-unit-preserving up to coherent natural isomorphism).

It may appear that, in the above definitions, we need $\mathcal{A}$ to be small and $\mathcal{V}$ or $\mathcal{C}$ to be cocomplete. We have written this way for ease of reading. Sometimes the necessary weighted (= "indexed") colimits exist for other reasons.

1 Proposition. *If $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a strong promonoidal functor then "existential quantification"*

$$\exists_\Phi : [\mathcal{A}, \mathcal{C}] \longrightarrow [\mathcal{B}, \mathcal{C}],$$

given by

$$\exists_\Phi(F)(B) = \int^{A} \mathcal{B}(\Phi A, B) \otimes FA,$$

has the structure of a strong monoidal functor.

Proof. Starting with the definitions of $\exists_\Phi$ and $*$, we have the calculation

$$\exists_\Phi(F * G)(B) = \int^{A} \mathcal{B}(\Phi A, B) \otimes \int^{A', A''} P(A', A'', A) \otimes (FA' \otimes GA'')$$

$$\cong \int^{A', A''} \int^{A} \mathcal{B}(\Phi A, B) \otimes P(A', A'', A) \otimes (FA' \otimes GA'')$$

by commuting colimits,

$$\cong \int^{A', A''} P(\Phi A', \Phi A'', B) \otimes (FA' \otimes GA'')$$

since Φ is strong,

$$\cong \int^{A', A''} \int^{B', B''} \mathcal{B}(\Phi A', B') \otimes \mathcal{B}(\Phi A'', B'') \otimes P(B', B'', B) \otimes (FA' \otimes GA'')$$

by the Yoneda Lemma,

$$\cong \int^{B',B''} P(B',B'',B)\otimes \int^{A'} \mathcal{B}(\Phi A',B')\otimes FA'\otimes \int^{A''} \mathcal{B}(\Phi A'',B'')\otimes GA''$$

by commuting colimits,

$$\cong (\exists_\Phi(F) * \exists_\Phi(G)(B) \qquad \text{by definitions.}$$

Similarly, we have

$$\exists_\Phi(J)(B) = \int^{A} \mathcal{B}(\Phi A, B)\otimes J(A) \cong J(B).$$

∎

The *cartesian* monoidal structure on a category with finite products has binary product as tensor product and the terminal object as unit. Dually, a category with finite coproducts has a *cocartesian* monoidal structure. If $\mathcal{A}$ is cocartesian monoidal and $\mathcal{C}$ is cartesian monoidal, then convolution on $[\mathcal{A},\ \mathcal{C}]$ is cartesian. Proposition 1 has the corollary that existential quantification $\exists_\Phi$ along a finite-coproduct-preserving functor Φ preserves finite products; compare [11], Proposition 2.7.

For any promonoidal category $\mathcal{A}$, the Yoneda embedding $Y : \mathcal{A} \longrightarrow [\mathcal{A},\mathcal{V}]^{op}$ is a promonoidal functor (just use the definition and the Yoneda Lemma). The closure in $[\mathcal{A},\mathcal{V}]^{op}$ of the representables $Y(A) = \mathcal{A}(A,-)$ under tensor products and unit (as in [4]) gives a full monoidal subcategory $\mathcal{A}'$ of $[\mathcal{A},\mathcal{V}]^{op}$, and Y factors through the inclusion via a promonoidal functor $N : \mathcal{A} \longrightarrow \mathcal{A}'$. This construction has a universal property: to describe it we introduce the ordinary category $PMon(\mathcal{A},\mathcal{B})$ whose objects are promonoidal functors $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ and whose arrows are promonoidal natural transformations ([2] and [5]); if $\mathcal{A}$, $\mathcal{B}$ are both monoidal, we write $Mon(\mathcal{A},\mathcal{B})$ for this same ordinary category. (Later we shall use the ordinary category $SPMon(\mathcal{A},\mathcal{B})$ of *strong* promonoidal functors.)

2 Proposition. *For each promonoidal category $\mathcal{A}$ and each monoidal category $\mathcal{B}$, restriction along $N : \mathcal{A} \longrightarrow \mathcal{A}'$ provides an equivalence of ordinary categories*

$$Mon(\mathcal{A}',\mathcal{B}) \xrightarrow{\sim} PMon(\mathcal{A},\mathcal{B}).$$

Proof. To see that restriction along N is essentially surjective, take a promonoidal functor $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$. We obtain the following diagram where regions commute up to canonical natural isomorphisms.

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{V}]^{op}} \arrow[rrr, "\exists_\Phi^{op}"] & & & {[\mathcal{B},\mathcal{V}]^{op}} \\
& \mathcal{A}' \arrow[ul] \arrow[r, dashed, "\Phi'"] & \mathcal{B}' \arrow[ur] & \\
\mathcal{A} \arrow[uu, "Y"] \arrow[ur, "N"'] \arrow[rrr, "\Phi"'] & & & \mathcal{B} \arrow[uu, "Y"'] \arrow[ul, "N"]
\end{tikzcd}$$

The functor $\exists_{\Phi}^{op}$ is monoidal. Thus, so is its restriction $\Phi' : \mathcal{A}' \longrightarrow \mathcal{B}'$. Since $\mathcal{B}$ is monoidal, the functor $N : \mathcal{B} \longrightarrow \mathcal{B}'$ is an equivalence of monoidal categories. So we obtain a promonoidal functor $\Psi : \mathcal{A}' \longrightarrow \mathcal{B}$ with $\Psi N \cong \Phi$. The remaining details are left to the reader; they will require the reader to know the definition of promonoidal natural transformation. ∎

Suppose $\mathcal{A}$ is a small promonoidal category. Observe that a strong promonoidal functor $\Phi : \mathcal{A} \longrightarrow \mathcal{C}^{op}$ satisfies the following conditions:

$$\int^{A''} P(A, A', A'') \otimes \mathcal{C}(B, \Phi A'') \xrightarrow{\cong} \mathcal{C}(B, \Phi A \otimes \Phi A')$$

$$\int^{A} JA \otimes \mathcal{C}(B, \Phi A) \xrightarrow{\cong} \mathcal{C}(B, I).$$

On tensoring both sides with B and using the Yoneda lemma, we obtain the conditions:

$$\int^{A''} P(A, A', A'') \otimes \Phi A'' \xrightarrow{\cong} \Phi A \otimes \Phi A'$$

$$\int^{A} JA \otimes \Phi A \xrightarrow{\cong} I.$$

Let $\mathcal{M} = SPMon(\mathcal{A}, \mathcal{C}^{op})^{op}$. There is a forgetful functor $\mathcal{M} \longrightarrow [\mathcal{A}^{op}, \mathcal{C}]$. The transform of a functor $F : \mathcal{A} \longrightarrow \mathcal{V}$ is the functor $\mathcal{T}(F) : \mathcal{M} \longrightarrow \mathcal{C}$ given by the coend

$$\mathcal{T}(F)(\Phi) = \int^{A} FA \otimes \Phi A \cong (\exists_{\Phi} F)(I).$$

Notice that this is the colimit of Φ weighted (or indexed) by F. We have defined a functor $\mathcal{T} : [\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{M}, \mathcal{C}]$. As usual, we regard $[\mathcal{A}, \mathcal{V}]$ as monoidal via convolution, but we regard $[\mathcal{M}, \mathcal{C}]$ as monoidal via pointwise tensor product in $\mathcal{C}$.

3 PROPOSITION *The transform enriches to a strong monoidal functor*

$$\mathcal{T} : [\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{M}, \mathcal{C}].$$

That is, the transform takes convolution to pointwise tensor product.

PROOF. For all $F, G : \mathcal{A} \longrightarrow \mathcal{V}$, we have the calculations

$$\begin{aligned}
\mathcal{T}(F * G)(\Phi) &= \int^{A} (F * G)(A) \otimes \Phi(A) \\
&\cong \int^{AA'A''} P(A', A'', A) \otimes F(A') \otimes G(A'') \otimes \Phi(A) \\
&\cong \int^{A', A''} F(A') \otimes G(A'') \otimes \Phi(A') \otimes \Phi(A'') \\
&\cong \mathcal{T}(F)(\Phi) \otimes \mathcal{T}(G)(\Phi)
\end{aligned}$$

$$\cong (\mathcal{T}(F)\otimes\mathcal{T}(G))(\Phi)$$

$$\mathcal{T}(J)(\Phi) = \int^A J(A)\otimes\Phi(A) \cong I.$$

∎

In particular, if $\mathcal{C}$ is cartesian closed, the transform takes convolution into cartesian product.

Remark One can also trace through the steps in [9] and obtain a generalisation to promonoidal structures using promonoidal functors.

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